Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 46(1) (2014) pp. 67- 75

A Comprehensive study of Pre A^*-functions

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Abstract. This manuscript is a study of Pre A^*-functions. Here a Pre A^*-function defined as a mapping $f : \mathbf{3}^n \longrightarrow \mathbf{3}$, where $\mathbf{3} = \{0, 1, 2\}$ is a Pre A^*-algebra. Further it has been determined various properties of Pre A^*-functions. Some basic properties of Pre A^*-functions such as duality, order relation and erstwhile properties are identified in this document.

AMS (MOS) Subject Classification Codes: 06E05, 06E25, 06E99, 06B10
Key Words: Pre A^*-algebra, Pre A^*-function, Pre A^*-variables, Pre A^*-expressions, duality and order relation

1. INTRODUCTION

Burris and Sankappanavar [1] made a detailed description on various aspects of boolean algebra. In a draft manuscript entitled ”The Equational theory of Disjoint Alternatives”, Manes [5] introduced the concept of Ada (Algebra of disjoint alternatives) $(A, \wedge, \vee, (-)', (-)_\pi, 0, 1, 2)$ which is however differs from the definition of the Ada of Manes [6] later paper entitled ”Adas and the equational theory of if-then-else”. While the Ada of the earlier draft seems to be based on extending the If-Then-Else concept more on the basis of Boolean algebras and the later concept is based on C-algebras $(A, \wedge, \vee, (-)^\sim)$

introduced by Fernando Guzman and Craig C. Squir [2]. Koteswara Rao [4] first introduced the concept of A*-algebra $(A, \wedge, \vee, *, (-)^{\sim}, (-)_{\pi}, 0, 1, 2)$ not only studied its equivalence with Ada, C-algebra, Ada's connection with 3-Ring, Stone type representation but also introduced the concept of A^*-clone, the If-Then-Else structure over A^*-algebra and Ideal of A^*-algebra. Venkateswara Rao [13] introduced the concept of Pre A^*-algebra $(A, \vee, , \wedge, (-)^{\sim})$ analogous to C-algebra as a reduct of A^*-algebra. Sadhan Kumar [10], Rechard [8], Kenneth [3] and Peter [7] described various aspects in the concept of boolean functions. Venkateswara Rao and Srinavasa Rao [14] studied about the concept of Pre A^*-Algebra as a poset Venkataswara Rao et al. [15] initiated a congruence relation and ternary operation on Pre A^*-Algebra.

Venkateswara Rao and Srinivasa Rao [11] defined the congruence relation on Pre A*-algebra. Venkateswara Rao and Srinivasa Rao [12], introduced the well known Cayley's theorem on centre of Pre A^*-algebras and also introduced an important operation on Pre A^*-algebra called ternary operation as $\Gamma(p, q) = (x \wedge p) \vee (x^{\sim} \wedge q)$.

Based on the definition and basic properties of Pre A^*-algebras and by combining and comparing properties of Boolean functions, in this manuscript there is defined a Pre A^*-function as a mapping $f : \mathbf{3}^n \longrightarrow \mathbf{3}$.

The first section is devoted to the introduction of Pre A^*-algebras and and various basic properties of Pre A^*-algebras.

The second section deals with the concept of Pre A^*-functions. So, this paper defines a Pre A^*-function as a mapping $f : \mathbf{3}^n \longrightarrow \mathbf{3}$, where $\mathbf{3} = \{0, 1, 2\}$ is a Pre A^*-algebra. Also, in this section, some important problems are given to more understanding of the notion of Pre A^*-functions.

The third section concerns on properties of Pre A^*-functions. Thus various basic properties of Pre A^*-functions such as duality, order relation and other properties are discussed in this paper.

2. Introduction to Pre A^*-algebras

Definition 1. An algebra $(A, \vee, \wedge, (-)^{\sim})$ where A is non-empty set with $\vee, \wedge$ are binary operations and $(-)^{\sim}$ is a unary operation satisfying the following axioms:

(1) $(x^{\sim})^{\sim} = x, \forall x \in A$;
(2) $x \wedge x = x, \forall x \in A$;
(3) $x \wedge y = y \wedge x, \forall x, y \in A$;
(4) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}, \forall x, y \in A$;
(5) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \forall x, y, z \in A$;
(6) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \forall x, y, z \in A$;
(7) $x \wedge y = x \wedge (x^{\sim} \wedge y), \forall x, y \in A$ is called a Pre A^*-algebra.

Example 1. $\mathbb{Z}_3 = \mathbf{3} = \{0, 1, 2\}$ with operations $\wedge, \vee, (-)^{\sim}$ defined as below is a Pre A^*-algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^\sim$
0	1
1	0
2	2

Note 1. The elements $0, 1, 2$ in the above example satisfy the following laws:
(a) $2^\sim = 2$ (b) $1 \wedge x = x$ for all $x \in \mathbf{3}$
(c) $0 \vee x = x$ for all $x \in \mathbf{3}$ (d) $2 \wedge x = 2 = 2 \vee x$ for all $x \in \mathbf{3}$.

Example 2. $\mathbb{Z}_2 = \mathbf{2} = \{0, 1\}$ with operations $\wedge, \vee, (-)\sim$ defined as below is a Pre A^*-algebra.

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

x	$x^\sim$
0	1
1	0

Note 2.

(1) $(\mathbf{2}, \vee, \wedge, (-)^\sim)$ is a Boolean algebra. So, every Boolean algebra is a Pre A^*-algebra.
(2) Axioms (i) and (iv) imply that the varieties of Pre A^*- algebras satisfy all the dual statements of (i) to (vii).

Theorem 2 ([9]). *Every Pre A^*-algebra satisfies the following laws.*

(1) $x \vee (x^\sim \wedge x) = x$
(2) $(x \vee x^\sim) \wedge y = (x \wedge y) \vee (x^\sim \wedge y)$
(3) $(x \vee x^\sim) \wedge x = x$
(4) $(x \vee y) \wedge z = (x \wedge z) \vee (x^\sim \wedge y \wedge z)$

3. PRE A*-FUNCTIONS:

This section deals with Pre A^*-functions and various examples of Pre A^*-functions. In this section, the binary operations $+$ and $\cdot$ are used in place of $\vee$ (meet) and $\wedge$ (join) respectively.

In section1, it is mentioned that $\mathbb{Z}_3 = \mathbf{3} = \{0, 1, 2\}$ is a Pre A^*-algebra. Now we define a Pre A^*-function on the Pre A^*-algebra $\mathbb{Z}_3$.

Note 3.1. A Pre A^*-variable is a variable which assumes only the values 0, 1 and 2. That is, it is a variable that takes values from $\mathbb{Z}_3$. Two Pre A^*-variables are said to be independent variables if they assume values from $\mathbb{Z}_3$ independent of each other. Clearly, the variables x and $x^\sim$ are not independent variables. If x_1 and x_2 are two independent Pre A^*-variables, then the ordered pair (x_1, x_2) assumes value from $\mathbb{Z}_3 \times \mathbb{Z}_3$ and the possible values assumed by (x_1, x_2) are $(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1)$ and $(2, 2)$. That is the ordered pair (x_1, x_2) has nine $(9 = 3^2)$ possible values.

Similarly, if x_1, x_2, x_3 are three independent Pre A^*-variables, then the ordered triplet (x_1, x_2, x_3) assumes value from $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and has $27 = 3^3$ possible values.

In general, if $x_1, x_2, \cdots, x_n$ are n independent Pre A^*-variables, the ordered n tuples $(x_1, x_2, \cdots, x_n)$ assumes value from $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3 = \mathbb{Z}_3^n$ and has 3^n possible values.

Definition 3. A mapping $f : \mathbb{Z}_3 \longrightarrow \mathbb{Z}_3$ is called a Pre A^*-function of one variable.

Note 3.2. From this, one can easily show that, there are 27 Pre A^*-functions of one variable.

Definition 4. A mapping $f : \mathbb{Z}_3^n \longrightarrow \mathbb{Z}_3$ is said to be a Pre A^*-function of n variables.

Note 3.3. As mentioned above, if $x_1, x_2, \cdots, x_n$ are n independent Pre A^*-variables, then the domain $\mathbb{Z}_3^n$ contains 3^n Pre A^* elements. For example, $\mathbb{Z}_3^2$ has 9 Pre A^*-variables, Z_3^3 has 27 Pre A^*-variables, Z_3^4 has 81 Pre A^*-variables, etc. So, consider a mapping $f : \mathbb{Z}_3 \longrightarrow \mathbb{Z}_3$.In $\mathbb{Z}_3$ there are $3 = 3^1$ number of elements. Thus from counting principle, the total number of Pre A^*-functions $f : \mathbb{Z}_3 \longrightarrow \mathbb{Z}_3$ is $3^{3^1} = 27$ (as mentioned above). For the mapping $f : \mathbb{Z}_3^2 \longrightarrow \mathbb{Z}_3$, in $\mathbb{Z}_3^2$ there are $9 = 3^2$ number of Pre A^*-variables, and the total number of Pre A^*-functions $f : \mathbb{Z}_3^2 \longrightarrow \mathbb{Z}_3$ is 3^{3^2}.

For the mapping $f : \mathbb{Z}_3^3 \longrightarrow \mathbb{Z}_3$, the total number of Pre A^*-functions is 3^{3^3}. In general by counting principle of products, the total number of Pre A^*-functions $f : \mathbb{Z}_3^n \longrightarrow \mathbb{Z}_3$ is 3^{3^n}.

Problem 3.1. Let x, y be two independent Pre A^*-variables and $f(x, y) = x + y^\sim$. Then find $f(0, 0), f(1, 2)$ and $f(2, 2)$.

Solution: Here f is a function $f : \mathbb{Z}_3^2 \longrightarrow \mathbb{Z}_3$ and x, y are independent Pre A^*-variables. Then; $f(0, 0) = 0 + 0^\sim = 0 + 1 = 1$ (Since $0^\sim = 1$)
$f(1, 2) = 1 + 2^\sim = 1 + 2 = 2$ (As $2^\sim = 2$)
$f(2, 2) = 2 + 2^\sim = 2 + 2 = 2$

Problem 3.2. Let x, y, z be three independent Pre A^*-variables and let $f(x, y, z) = xy + xy^\sim + z^\sim$. Then find $f(1, 0, 2), f(0, 2, 2)$ and $f(1, 1, 1)$.

Solution: In similar fashion with problem 2.1 above, where $f : \mathbb{Z}_3^3 \longrightarrow \mathbb{Z}_3$, we have;
$f(1, 0, 2) = 1 \cdot 0 + 1 \cdot 0^\sim + 2^\sim = 0 + 1 \cdot 1 + 2 = 0 + 1 + 2 = 2$
$f(0, 2, 2) = 0 \cdot 2 + 0 \cdot 2^\sim + 2^\sim = 2 + 0 \cdot 2 + 2 = 2 + 2 + 2 = 2$
$f(1, 1, 1) = 1 \cdot 1 + 1 \cdot 1^\sim + 1^\sim = 1 + 1 \cdot 0 + 0 = 1 + 0 + 0 = 1$

Note 3.4. From the above two examples, we have an interesting property of Pre A^*-functions.

Theorem 5. *If any Pre A^*-variable assumes the value 2 in its Pre A^*-function (that is, in its functional value), then the function has the value 2.*

Proof. Without loss of generality, let $f : \mathbb{Z}_3^3 \longrightarrow \mathbb{Z}_3$ be a Pre A^*-function such that $f(x, y, z) = xy^\sim + xy + yz^\sim + xz$. Suppose the variable y assumes the value 2 (that is $y = 2$), then; $f(x, 2, z) = x \cdot 2^\sim + x \cdot 2 + 2 \cdot z^\sim + xz = x \cdot 2 + x \cdot 2 + 2 \cdot z^\sim + xz$ (Since $2^\sim = 2$) $= 2 + 2 + 2 + xz = 2 + xz = 2$ (By the definition of Pre A^*-algebra, $x + 2 = x \cdot 2 = 2, \forall x \in \mathbb{Z}_3$.) □

Note 3.5. This property does not hold in the case of Boolean functions. Though $x + 1 = 1$,$\forall x \in$ Boolean algebra B but $x \cdot 1 = x, \forall x \in B$.

Note 3.6. Let f be a Pre A^*-function. Then $f(x) = x + x^\sim$ and $f(x) = xx^\sim$ are in their simplified form because, in a Pre A^*-algebra the properties $x + x^{'} = 1$ and $xx^{'} = 0$ do not hold in general. But in the case of Boolean function $f(x) = x + x^{'} = 1$ and $f(x) = xx^{'} = 0$ are in their simplified form (Since $x + x^{'} = 1, xx^{'} = 0, \forall x \in B$.)

Problem 3.3. Simplify the Pre A^*-function $f(x, y, z) = xyz + xyx^{\sim}z + xyzy^{\sim}$.
Solution: f is a Pre A^*-function $f : \mathbb{Z}_3^3 \longrightarrow \mathbb{Z}_3$ and $f(x, y, z) = xyz + xyx^{\sim}z + xyzy^{\sim}$ $= xyz + xx^{\sim}yz + xyy^{\sim}z$ (Since $xy = yx$) $= (x + xx^{\sim})yz + xyy^{\sim}z = xyz + xyzy^{\sim}$ (Since $x + xx^{\sim} = x$) $= x(y + yy^{\sim})z = xyz$.

Problem 3.4. Simplify the Pre A^*-function $f(x, y, z) = xy(z+z^{\sim})z+xyzy^{\sim}+xz^{\sim}$.
Solution: $f(x, y, z) = xyzz + xyz^{\sim}z + xyzy^{\sim} + xz^{\sim} = xyz + xyzz^{\sim} + xyy^{\sim}z + xz^{\sim}$ (Since $zz = z$) $= xy(z + zz^{\sim}) + xyy^{\sim}z + xz^{\sim} = xyz + xyy^{\sim}z + xz^{\sim}$ (As $z + zz^{\sim} = z$) $= x(y + yy^{\sim})z + xz^{\sim} = xyz + xz^{\sim}$ (As $y + yy^{\sim} = y$)
Problem 3.5. Show that $f(x, y) = xy + xyx^{\sim} + xy + yxy^{\sim} = xy$.
Solution: $f(x, y) = (x + xx^{\sim})y + xy + xyy^{\sim} = xy + x(y + yy^{\sim}) = xy + xy = xy$.
From the above problems, one can observe that, a Boolean function can be simplified into more simplified form than a Pre A^*-function and a Boolean function is easy to simplify than a Pre A^*-function. For instance, the Pre A^*-function $f(x, y, x) = xyzy^{\sim}$ is in its simplified form. But the Boolean function $f(x, y, z) = xyzy'$ is not in its simplified form. Since if $f(x, y, z) = x(yy^{'})z = x(0)x = 0$ (Since $x \cdot 0 = 0, \forall x \in B$).
Note 3.7: Variables of a Boolean function can be taken as propositional variables. Because, Boolean algebra itself is the study of logic, and a proposition is a declarative sentence which has a truth value of true or false but not both.

Similarly, each Boolean variable has the value 0 or 1 but not both and we can associate the truth value true by 1 and the truth value false by 0. But a Pre A^*-function is an extension of this function, and introduces another proposition with undefined truth value that can be represented by the value 2.

4. Properties of Pre A^*-functions

In this section we give attention to various basic properties of Pre A^*-functions. A Pre A^*-expression in the variables $x_1, x_2, \cdots, x_n$ are defined recursively as $0, 1, 2, x_1, x_2, \cdots,$ x_n are Pre A^*-expressions. If E_1 and E_2 are Pre A^*-expressions then $E_1^{\sim}, (E_1 + E_2)$ and (E_1E_2) are also Pre A^*-expressions. Each Pre A^*-expression represents a Pre A^*-function.

Definition 6. Let f be a Pre A^*-function, then the algebraic degree of f denoted by $deg(f)$ is the number of variables in the highest order term.

Example 4.1. The function $f(x) = 1$ has degree zero.
The function $f(x) = x$ has degree one.
The function $f(x, y) = x + xy$ has degree two.
The function $f(x, y, z) = x + xz + xyz$ has degree three.

Definition 7. The dual of a Pre A^*-expression is obtained by interchanging Pre A^*-sums and Pre A^*-products, interchanging 0s and 1s and interchanging of 2 with itself.

Example 4.2. The dual of the Pre A^*-expression $x(y + 0)$ is $x + (y \cdot 1)$ which is also Pre A^*-expression. The dual of $x^{\sim} \cdot 2 + (y^{\sim} + z)$ is $x^{\sim} + 2 \cdot (y^{\sim} \cdot z)$.
Note 4.1. The dual of a Pre A^*-function f is represented by a Pre A^*-expression is a function represented by the dual of this expression, and is denoted by f^d.
An identity between Pre A^*-functions remain valid when the dual of both sides of the identity are taken. This is called the principle of duality, and is useful for obtaining new identity.

Example 4.3.. By taking the duality on both sides of the identity $x + (xx^{\sim}) = x$, we obtain the identity $x \cdot (x + x^{\sim}) = x$.

Theorem 8. *Let $f : \mathbb{Z}_3^n \longrightarrow \mathbb{Z}_3$ be any Pre A^*-function, then the following holds;*
a) $f + 2 = 2 = 2 + f$
b) $f \cdot 2 = 2 = 2 \cdot f$

Proof. Since $f : \mathbb{Z}_3^n \longrightarrow \mathbb{Z}_3$ is any Pre A^*-function, its value is an element of $\mathbb{Z}_3 = \{0, 1, 2\}$. Hence, from the definition of Pre A^*-algebra; $x + 2 = 2 = 2 + x$ for all $x \in \mathbb{Z}_3$ $x \cdot 2 = 2 = 2 \cdot x$ for all $x \in \mathbb{Z}_3$ (As $\mathbb{Z}_3$ is a Pre A^*-algebra).
Consequently, (a) and (b) follows. This completes the proof. □

Definition 9. Let f and g be two Pre A^*-functions of degree n. The sum $f + g$ (Pre A^*-sum) and the Pre A^*-product fg are defined as; $(f + g)(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n) + g(x_1, x_2, \cdots, x_n)$ and $(fg)(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n)g(x_1, x_2, \cdots, x_n)$.

Definition 10. The Pre A^*-functions f and g of n variables are said to be equal if and only if $f(x_1, x_2, \cdots, x_n) = g(x_1, x_2, \cdots, x_n)$.

Definition 11. The dual of a Pre A^*-function f is the function f^d defined by $f^d(X) = [f(X^{\sim})]^{\sim} \forall X = (x_1, x_2, \cdots, x_n) \in \mathbf{3}^n$,where $X^{\sim} = (x_1^{\sim}, x_2^{\sim}, \cdots, x_n^{\sim})$.

Example 4.4:. Let f be the two variable Pre A^*-function defined by $f(0, 0) = 1, f(0, 2) = 2, f(1, 1) = 1, f(0, 1) = 1, f(1, 2) = 2$ and $f(1, 0) = 0$. Find f^d.
Solution: $f^d(0, 0) = [f(0^{\sim}, 0^{\sim})]^{\sim} = [f(1, 1)]^{\sim} = 1^{\sim} = 0$
$f^d(0, 2) = [f(0^{\sim}, 2^{\sim})]^{\sim} = [f(1, 2)]^{\sim} = 2^{\sim} = 2$
$f^d(1, 1) = [f(1^{\sim}, 1^{\sim})]^{\sim} = [f(0, 0)]^{\sim} = 1^{\sim} = 0$
$f^d(0, 1) = [f(0^{\sim}, 1^{\sim})]^{\sim} = [f(1, 0)]^{\sim} = 0^{\sim} = 1$
$f^d(1, 2) = [f(1^{\sim}, 2^{\sim})]^{\sim} = [f(0, 2)]^{\sim} = 2^{\sim} = 2$
$f^d(1, 0) = [f(1^{\sim}, 0^{\sim})]^{\sim} = [f(0, 1)]^{\sim} = 1^{\sim} = 0$.

Theorem 12. *If f and g are two Pre A^*-functions, then the following holds.*

(1) $(f^d)^d = f$ *(Involution: the dual of the dual is the function itself)*
(2) $(f^{\sim})^d = (f^d)^{\sim}$
(3) $(f + g)^d = f^d g^d$
(4) $(fg)^d = f^d + g^d$

Proof. (a) and (b) follow immediately from the definition of duality. For property (c), consider; $(f + g)^d(X) = (f + g)^{\sim}(X^{\sim}) = [f(X^{\sim}) + g(X^{\sim})]^{\sim} = [f(X^{\sim})]^{\sim}[g(X^{\sim})]^{\sim}$ (By De Morgan's law) $= f^d g^d$ Property (d) follows from the properties (a) and (c). □

Note 4.2. A unary operation $* : x \longrightarrow x^*$ on a non empty set A is called an involution if $(x^*)^* = x, \forall x \in A$.

Corollary 13. *If we define the Pre A^*-function 2 by $2(X) = 2, \forall X \in 3^n$, then $(f + 2)^d = 2 = (f \cdot 2)^d$.*

Proof. $(f + 2)^d(X) = (f + 2)^{\sim}(X^{\sim}) = [(f + 2)(X^{\sim})]^{\sim} = [f(X^{\sim})]^{\sim}[2(X^{\sim})]^{\sim}$ (By property (c) above) $= f^d \cdot 2^{\sim}$ (By the definition of 2) $= 2$ (By theorem 8 above) In a similar fashion, we have $(f \cdot 2)^d = 2 \Longrightarrow (f + 2)^d = 2 = (f \cdot 2)^d$. □

Definition 14. Let f be a Pre A^*-function of degree n. Then $f^{\sim}$ is a Pre A^*-function and is defined as $f^{\sim}(x_1, x_2, \cdots, x_n) = [f(x_1, x_2, \cdots, x_n)]^{\sim}$.

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Definition 15. The relation $\leq$ on the set of Pre A^*-functions of degree n is defined as $f \leq g$, where f and g are Pre A^*-functions if and only if; $g(x_1, x_2, \cdots, x_n) = 2$ whenever $f(x_1, x_2, \cdots, x_n) = 2$.

Example 4.5. Let f and g be two Pre A^*-functions such that $f(x, y) = x$ and $g(x, y) = x + y$. Then $f \leq g$.
Solution: Let $f(x, y) = x = 2$ which implies that $x = 2$. Then, $g(x, y) = x + y = 2 + y = 2 \forall y \in \mathbb{Z}_3$. Which implies that if $f = 2$ then $g = 2$. Therefore $f \leq g$.

Theorem 16. *If f and g are Pre A^*-functions of degree n then follows the following.*
a) $f \leq f + g$
b) $fg \leq f$

Proof. (a) Let f and g be Pre A^*-functions of degree n. If $f(x_1, x_2, \cdots, x_n) = 2$ then $(f+g)(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n)+g(x_1, x_2, \cdots, x_n) = 2+g(x_1, x_2, \cdots, x_n) = 2$ (By the dominance property of 2) Hence $f(x_1, x_2, \cdots, x_n) = 2 \Longrightarrow (f+g)(x_1, x_2, \cdots, x_n) = 2$. Therefore $f \leq f + g$. (b)Let $(fg)(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n)g(x_1, x_2, \cdots, x_n) = 2$. Hence $f(x_1, x_2, \cdots, x_n) = 2$. Which implies that, $fg \leq f$. □

Note 4.3. From the above theorem 3.2, it is also true that $g \leq f + g$ and $fg \leq g$.

Theorem 17. *The relation $\leq$ is a partial ordering on the set of Pre A^*-functions of degree n.*

Proof. Let f, g and h be Pre A^*-functions of order n. Then $f(x_1, x_2, \cdots, x_n) = 2 \Longrightarrow f(x_1, x_2, \cdots, x_n) = 2$ is reflexive. Suppose that $f \leq g$ and $g \leq h$ then, $f(x_1, x_2, \cdots, x_n) = 2$ if and only if $g(x_1, x_2, \cdots, x_n) = 2$ which implies that $f = g$. Thus $\leq$ is anti symmetric. Assume that $f \leq g \leq h$, then if $f(x_1, x_2, \cdots, x_n) = 2$, it follows that $g(x_1, x_2, \cdots, x_n) = 2$, which implies that $h(x_1, x_2, \cdots, x_n) = 2$. That is $f(x_1, x_2, \cdots, x_n) = 2 \Longrightarrow h(x_1, x_2, \cdots, x_n) = 2 \Longrightarrow f \leq h$. Hence the relation $\leq$ is transitive. Therefore, the relation $\leq$ is a partial order on the set of Pre A^*-functions. □

Definition 18. A join semi lattice $(S, \vee)$ is said to be directed above if and only if for $x, y \in S$, there exists an element $a \in S$ such that $a \geq x, a \geq y$.

Theorem 19. *Let F be the set of all Pre A^*-functions. Then $(F, \vee)$ is a directed above join semi lattice. But $(F, \wedge)$ is not a meet semi lattice.*

Proof. Define $(f \vee g)(X) = f(X) \vee g(X), (f \wedge g)(X) = f(X) \wedge g(X), \forall x \in \mathbb{Z}_3^n$, where f and g are Pre A^*-functions from $\mathbb{Z}_3^n \longrightarrow \mathbb{Z}_3, f^{\sim}(X) = [f(X)]^{\sim}, 0(X) = 0, 1(X) = 1, 2(X) = 2\ \forall X \in \mathbb{Z}_3^n$. Then we have that; $[(f \vee g) \vee h](X) = (f \vee g)(X) \vee h(X) = [f(X) \vee g(X)] \vee h(X) = f(X) \vee [g(X) \vee h(X)] = f(X) \vee [(g \vee h)(X)] = [f \vee (g \vee h)](X)$ (The associative property of $\vee$ is simply inherited from the definition of Pre A^*-algebra.) $(f \vee f)(X) = f(X) \vee f(X) = f(X), \forall f \in F$ (Since $x \vee x = x, \forall x \in 3$). Hence $(F, \vee)$ is a join semi lattice. For all $f, g \in F$ there is a function $2 = 2(X), \forall X \in \mathbf{3}^n$ such that $2 \geq f, 2 \geq g$. (Since in a Pre A^*-function $2 \vee f = f \vee 2 = 2 + f = 2, \forall f \in F$.) For all $f, g \in F, f \vee g = g \vee f$. Therefore, $(F, \vee)$ is a directed above join semi lattice. But $(F, \vee)$ is not a meet semi lattice. If, let $f(x, y) = x \vee y$ be a Pre A^*-function from $\mathbf{3}^2$ to $\mathbf{3}^2$ then; $[f(x, y)] \wedge [f(x, y)] = (x \vee y) \wedge (x \vee y) = (x \wedge x) \vee (x \wedge y) \vee (x \wedge y) \vee (y \wedge y) = x \vee (x \wedge y) \vee y = x \vee (x \vee 1) \wedge y \neq x \vee y = f(x, y)$ (Since $x \vee 1 \neq 1, \forall x \in \mathbf{3}$). Which implies that $f \vee f \neq f, \forall f \in F$. Thus $(F, \wedge)$ is not meet semi lattice. □

Note 4.4.

(1) Let F denotes the set of all Boolean functions from $\mathbb{Z}_2^n \longrightarrow \mathbb{Z}_2$. Define $(f \vee g)(X) = f(X) \vee g(X), (f \wedge g)(X) = f(X) \wedge g(X), \forall X \in \mathbb{Z}_2^n$
$f^{'}(X) = [f(X)]^{'}, 0(X) = 0, 1(X) = 1, \forall X \in \mathbb{Z}_2^n$. Then the set $(F, \vee, \wedge)$ forms a lattice. But the set of Pre A^*-functions does not form a lattice under these two binary operations. Because, the property $x = x \vee (x \wedge y)$ and its dual (absorption laws) and the idempotent law $x \wedge x = x$ for a set to be a lattice do not hold on the set of Pre A^*-functions.

(2) $(F, \vee, \wedge)$, where F is the set of Boolean functions, is a complemented lattice. But not the set of Pre A^*-functions.

Note 4.5. A bounded lattice L is said to be a complemented lattice if for each $a \in L$ there exists an element $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$.

Conclusion: It is observed that in general, if $x_1, x_2, \cdots, x_n$ are n independent Pre A^*-variables, the ordered n tuples $(x_1, x_2, \cdots, x_n)$ assumes value from $\mathbb{Z}_3 \times \mathbb{Z}_3 \cdots \times \mathbb{Z}_3 = \mathbb{Z}_3^n$ and has 3^n possible values. It is concluded that, there are 27 Pre A^*-functions of one variable. Also in general by counting principle of products, it is obtained the total number of Pre A^*-functions $f : \mathbb{Z}_3^n \longrightarrow \mathbb{Z}_3$ is 3^{3^n}. It is noticed that if any Pre A^*-variable assumes the value 2 in its Pre A^*-function (that is in its functional value), then the function has the value 2. It has been observed that, a Boolean function can be simplified into more simplified form than a Pre A^*-function and a Boolean function is easy to simplify than a Pre A^*-function. The principle of duality of a Pre A^*-expression is obtained. An identity between Pre A^*-functions remain valid when the dual of both sides of the identity are taken. Also there is defined the relation $\leq$ and verified that the relation $\leq$ is a partial order on the set of Pre A^*-functions. It is observed that the set of Boolean functions form a lattice and the set of Pre A^*-functions does not form a lattice under these two binary operations. It is observed that the set of all Pre A^*-functions is a directed above join semi lattice but is not a meet semi lattice.

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